A UNITAL VERSION OF KK-THEORY

JOINT WORK WITH J. GABE AND C. SCHAFHAUSER

José Carrión
TCU

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Classification theorem (many hands)

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An essential tool for this classification is KK-theory.

Today: KK_u, a "KK that remembers the unit", and why we'd want it.

A unital separable, B σ-unital; $\mathcal{K}$ = compacts, $\mathcal{M}(B \otimes \mathcal{K})$ = multiplier algebra of $B \otimes \mathcal{K}$.

Cuntz pair $(\varphi, \psi) \colon A \rightrightarrows \mathcal{M}(B \otimes \mathcal{K})$

two *-homomorphisms into $\mathcal{M}(B \otimes \mathcal{K})$ agreeing modulo the compacts: $\varphi(a) - \psi(a) \in B \otimes \mathcal{K}$ for all $a \in A$

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- think: "formal difference" $[\varphi] - [\psi]$ of two representations
- $KK(\mathbb{C}, B) \cong K_0(B), \quad KK^1(\mathbb{C}, B) \cong K_1(B)$

THE TWO RELATIONS

Compare $\varphi, \psi : A \to \mathcal{M}(B \otimes \mathcal{K})$ (a Cuntz pair) by conjugating unitaries in $(B \otimes \mathcal{K})^{\sim}$, with $\|u_\bullet\, \varphi(a)\, u_\bullet^* - \psi(a)\| \to 0$:

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- approximate: a *sequence* $(u_n)_{n\in\mathbb{N}}$ $\rightsquigarrow KL(A, B)$
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KK/KL-uniqueness (Szabó)

For absorbing φ, ψ (roughly: each contains a copy of every representation):

$[\varphi, \psi] = 0$ in $KL(A, B)$ $\iff$ φ, ψ approximately u.e.

$[\varphi, \psi] = 0$ in $KK(A, B)$ $\iff$ φ, ψ asymptotically u.e. via a path with $u_0 = 1$

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This distinction has a long history in Ext-theory. BDF classified *unital* extensions

$$0 \to \mathcal{K} \to E \to C(X) \to 0.$$

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For general B there is a difference: KK vs. KK_u.

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$$\pi_n\big(\operatorname{Aut}(A)\big) \;\cong\; KK_u^n(A, A), \qquad n \geq 1.$$

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$$\pi_n\big(\mathrm{Aut}(A)\big) \;\cong\; KK_u^n(A,A), \qquad n \geq 1.$$

Note that KK cannot separate φ from $\mathrm{Ad}(w)\varphi$, yet $\mathrm{Ad}(w)$ need not connect to the identity in $\mathrm{Aut}(A)$.
Want KK_u to remember w (really, $[w]_1 \in K_1(A)$).

Definition (C–Gabe–Schafhauser)

A Cuntz pair (φ, ψ) is *unital* if $\varphi(1_A) = \psi(1_A) = 1_{\mathcal{M}(B\otimes\mathcal{K})}$.

$$KK_u(A, B) := \{\text{unital Cuntz pairs}\} \,/\, \text{unital homotopy}$$

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Similar to *KK*:

- abelian group: $[\varphi_1, \psi_1]_u + [\varphi_2, \psi_2]_u := [\varphi_1 \oplus \varphi_2, \psi_1 \oplus \psi_2]_u$
- $-[\varphi, \psi]_u = [\psi, \varphi]_u$; identity element $[\varphi, \varphi]_u$

Forgetting that a unital Cuntz pair is unital gives

$$\cancel{u} : KK_u(A, B) \longrightarrow KK(A, B).$$

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$$\not{u} : KK_u(A, B) \longrightarrow KK(A, B).$$

Questions

- kernel? (what does KK_u see that KK doesn't?)
- image? (which KK-classes have unital representatives?)
- is KK_u computable?

The forgetful map fits into a five-term exact sequence:

$$KK^1(A,B) \xrightarrow{v^*} K_1(B) \xrightarrow{\partial_u} KK_u(A,B) \xrightarrow{\not{u}} KK(A,B) \xrightarrow{v^*} K_0(B)$$

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- $v\colon \mathbb{C} \to A$: the unital inclusion
- the boundary map is explicit: $\partial_u[w]_1 = [\pi,\, \mathrm{Ad}(w)\pi]_u$, where $\pi\colon A \to \mathcal{M}(B \otimes \mathcal{K})$ is “unitally” absorbing

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- image of $\not u$ = kernel of $KK(A,B) \xrightarrow{v^*} K_0(B)$
  (a KK-class has a unital representative iff it's compatible with the units)
- computability: from $KK(A,B)$, $K_*(B)$, and the maps you get $KK_u(A,B)$ up to a group extension

The exact sequence is obtained alongside the following isomorphism. The mapping cone of the unital inclusion v is

$$C_v A := \left\{ f \in C_0([0,1), A) : f(0) \in \mathbb{C}1_A \right\},$$

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Theorem (C–Gabe–Schafhauser)

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The five-term sequence is its Puppe sequence; homotopy invariance, stability, and a product $KK_u \times KK \to KK_u$ come with it.

Classical: $KK^1(A,B) \cong \mathrm{Ext}^{-1}(A,B)$.

Theorem (C–Gabe–Schafhauser; cf. Skandalis '91)

$KK^1_u(A,B) \cong \mathrm{Ext}_u^{-1}(A,B)$

The unital and non-unital Ext groups differ by a $K_1(B)$ term, invisible for $B = \mathcal{K}$.

EXAMPLE: $A = \mathcal{O}_2$

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So the exact sequence collapses to

$$0 \to K_1(B) \xrightarrow{\partial_u} KK_u(\mathcal{O}_2, B) \to 0, \qquad \text{i.e.} \qquad KK_u(\mathcal{O}_2, B) \cong K_1(B).$$

WHICH UNITARIES?

Back to comparing unital maps in a Cuntz pair $(\varphi, \psi)\colon A \rightrightarrows \mathcal{M}(B \otimes \mathcal{K})$, up to asymptotic unitary equivalence.

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It's important that unitaries conjugating φ toward ψ don't disturb that shared quotient. The clean way: require

$$u_t \in 1 + B \otimes \mathcal{K} \subseteq \mathcal{M}(B \otimes \mathcal{K})$$

so the image of u_t in the quotient is a scalar.

(This requirement gives "proper" asymptotic unitary equivalence.)

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Suppose *some* path (u_t) implements $\varphi \sim \psi$.

Want to go from 1 to u_0 first, then follow the path. Possible iff u_0 connects to 1 in $\mathcal{U}((B \otimes \mathcal{K})^{\sim})$:

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KK_u decides proper strong asymptotic equivalence for unital maps.

Theorem (C–Gabe–Schafhauser)

Let $\varphi, \psi : A \to \mathcal{M}(B \otimes \mathcal{K})$ be unitally absorbing. Then

$$[\varphi, \psi]_u = 0 \text{ in } KK_u(A, B)$$

if and only if there is a norm-continuous path $(u_t)_{t \geq 0}$ in $\mathcal{U}((B \otimes \mathcal{K})^\sim)$ with $u_0 = 1$ and

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KK_u-classes decide proper strong asymptotic unitary equivalence, the *unital analog* of Szabó's KK-uniqueness, via his K_1-injectivity.

The *Paschke dual* of $\pi\colon A \to \mathcal{M}(B \otimes \mathcal{K})$ is

$$\mathcal{D}_\pi = \{x \in \mathcal{M}(B \otimes \mathcal{K}) : [x, \pi(a)] \in B \otimes \mathcal{K} \ \ \forall\, a \in A\}.$$

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If π is unitally absorbing, then

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- *existence* built in: every class is $[\pi, \mathrm{Ad}(U)\pi]_u$
- uses: Szabó's K_1-injectivity theorem for Paschke duals, plus a Gabe–Szabó lemma

THANK YOU!